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ON A NEW SUBCLASS OF MEROMORPHIC HARMONIC FUNCTIONS WITH FIXED RESIDUE α

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Abstract. In this paper, we obtain some properties such as coefficient conditions, distortion theorem and extreme points for a certain subclass of meromorphic harmonic functions with fixed residue α, $0 \leq \alpha < 1$.

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1. Introduction

The important study initiated by Clunie and Sheil-Small [3] on the class H consisting of complex valued harmonic sense-preserving univalent functions f in a simply connected domain $D \subseteq C$ defined on the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f_z(0) - 1 = 0$ formed the basis for various studies related to different subclasses of harmonic univalent functions. It is known that [3] each function $f \in H$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in D. In fact H reduces to g, the class of normalized univalent functions if the co-analytic part of f is zero. A compact survey on harmonic univalent functions is given by Ahuja [1].

Jinxi Ma [8] considered the class S_p of functions f which are meromorphic and univalent in the unit disc Δ normalized by $f(0) = 0$, $f'(0) = 1$ and $f(p) = \infty$, with $0 < p < 1$.

Observe that in the annulus $\{z : p < |z| < 1\}$ each function h in S_p admits an expansion written as

$$h(z) = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n \qquad (1)$$

where $\alpha = Res(f, p)$ with $0 < \alpha \leq 1$, $z \in \Delta \backslash \{p\}$.

Jinxi Ma [8] and also Ghanim and Darus [4, 6] have made use of the function h given in (1) and studied some properties.

Let HS_p denote the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the punctured unit disk $\Delta\backslash\{p\}$.

For $f = h + \bar{g}$, we may write the analytic function h as in (1) and g as

$$g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Then we have

$$f(z) = h(z) + \overline{g(z)} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \tag{2}$$

where $\alpha = Res(f,p)$, with $0 < \alpha \leq 1$, $z \in \Delta\backslash\{p\}$.

Let $\overline{HS}_p$ be a subclass of HS_p consisting of function of the form

$$f(z) = h(z) + \overline{g(z)} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad (a_n, b_n \geq 0) \tag{3}$$

where $\alpha = Res(f,p)$, with $0 < \alpha \leq 1$, $z \in \Delta\backslash\{p\}$, which are univalent harmonic in the punctured unit disk $\Delta\backslash\{p\}$. The functions h and g are analytic in $\Delta\backslash\{p\}$ and Δ respectively and h has a simple pole at the point p with residue α.

For $\alpha = 1$ and $p = 0$, the function f defined in (3) was studied by Bostanci, Yalcin and Öztürk [2].

In [4, 5, 6] Ghanim et al. defined the operator I^k on the class HS_p as follows:

$$\begin{aligned} &I^0 f(z) = f(z), \\ &I^k f(z) = I^k h(z) + \overline{I^k g(z)}, \quad k = 1, 2, 3, \ldots, \end{aligned} \tag{4}$$

where,

$$I^k h(z) = z(I^{k-1}h(z))' + \frac{\alpha(2z-p)}{(z-p)^2} = \frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n$$

and

$$I^k g(z) = z(I^{k-1}g(z))' = \sum_{n=1}^{\infty} n^k b_n z^n.$$

Motivated by the earlier works of [2, 6, 7, 9], we now introduce a new subclass $HS_p^*(k, \alpha, \beta, \mu)$ using the differential operator I^k.

Definition 1. *A function $f \in HS_p^*(k,\alpha,\beta,\mu)$, if it satisfies*

$$\left|\frac{H(z)}{\mu H(z)+(1-\mu)}+1\right| \le \left|\frac{H(z)}{\mu H(z)+(1-\mu)}+2\beta-1\right| \tag{5}$$

where $H(z) = \dfrac{z(I^k h(z))' + \overline{z(I^k g(z))'}}{I^k f(z)}$, $(k \in N_0 = N \cup \{0\})$, $0 \le \beta < 1$, $0 \le \mu < 1$ *and for all z in $\Delta \backslash \{p\}$.*

Remark 1. $HS_p^*(k,\alpha,\beta,0) = SH_p^*(k,\alpha,\beta)$ *[6].*

Also

$$HS_p^*[k,\alpha,\beta,\mu] = HS_p^*(k,\alpha,\beta,\mu) \cap \overline{HS}_p.$$

We now obtain the coefficient estimates for the classes $HS_p^*(k,\alpha,\beta,\mu)$ and $HS_p^*[k,\alpha,\beta,\mu]$.

2. Main Results

A sufficient coefficient condition for functions analytic in $\Delta\backslash\{p\}$ to be in $HS_p^*(k,\alpha,\beta,\mu)$ is now derived.

Theorem 1. *Let $f(z) = h(z) + \overline{g(z)}$ be given by (2).*
If

$$\sum_{n=1}^{\infty} n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2(|a_n|+|b_n|) \le \alpha(1-\beta)[(1-p)-\mu(2-p)], \tag{6}$$

where $0 \le \beta < 1$, $k \in N_0$, then f is sense preserving in $\Delta\backslash\{p\}$ and $f \in HS_p^*(k,\alpha,\beta,\mu)$.

Proof. Assume that (6) holds true for $0 \le \beta < 1$. Then by (5) we have

$$\left|\frac{H(z)}{\mu H(z)+(1-\mu)}+1\right| < \left|\frac{H(z)}{\mu H(z)+(1-\mu)}+2\beta-1\right|$$

This gives

$$\left|[z(I^k h(z))' + \overline{z(I^k g(z))'}](1+\mu) + I^k f(z)(1-\mu)\right| < \left|[z(I^k h(z))' + \overline{z(I^k g(z))'}](1+\mu(2\beta-1)) + (2\beta-1)(1-\mu)I^k f(z)\right|.$$

Let

$$\begin{aligned} M(z) &= \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1+\mu) + I^k f(z)(1-\mu) \right| \\ &\quad - \left| [z(I^k h(z))' + \overline{z(I^k g(z))'}](1+\mu(2\beta-1)) + (2\beta-1)(1-\mu) I^k f(z) \right|. \end{aligned}$$

Then, for $|z| = r$, and since $|z-p| \geq |z| - p = r - p$, we have

$$\begin{aligned} M(z) = &\left| (1+\mu)\left[-\frac{\alpha z}{(z-p)^2} + z\sum_{n=1}^{\infty} n^{k+1} a_n z^{n-1} + z\sum_{n=1}^{\infty} n^{k+1} \overline{b_n z^{n-1}} \right] \right. \\ &\left. + (1-\mu)\left[\frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n + \sum_{n=1}^{\infty} n^k \overline{b_n z^n} \right] \right| \\ &- \left| (1+\mu(2\beta-1))\left[-\frac{\alpha z}{(z-p)^2} + z\sum_{n=1}^{\infty} n^{k+1} a_n z^{n-1} + z\sum_{n=1}^{\infty} n^{k+1} \overline{b_n z^{n-1}} \right] \right. \\ &\left. + (2\beta-1)(1-\mu)\left[\frac{\alpha}{z-p} + \sum_{n=1}^{\infty} n^k a_n z^n + \sum_{n=1}^{\infty} n^k \overline{b_n z^n} \right] \right|. \end{aligned}$$

Also we notice that

$$\begin{aligned} M(r) \leq &\frac{\alpha p + \mu(2\alpha r - \alpha p)}{(r-p)^2} + \sum_{n=1}^{\infty} n^k [(n+1) + \mu(n-1)]\ [|a_n| r^n + |b_n| r^n] \\ &- \frac{[2\alpha r - 2\alpha\beta r + 2\alpha\beta p - \alpha p + \mu[4\alpha\beta r - 2\alpha r - 2\alpha\beta p + \alpha p]]}{(r-p)^2} \\ &+ \sum_{n=1}^{\infty} n^k [(n+2\beta-1) + \mu(2\beta-1)(n-1)]\ [|a_n| r^n + |b_n| r^n] \\ &= -\frac{2\alpha(1-\beta)}{r-p} + \frac{2\alpha\mu(2r-p)(1-\beta)}{(r-p)^2} \\ &+ \sum_{n=1}^{\infty} n^k [2(n+\beta) + 2\beta(n-1)\mu]\ [|a_n| + |b_n|] r^n. \end{aligned}$$

In other words

$$\begin{aligned} (r-p)^2 M(r) \leq &\sum_{n=1}^{\infty} n^k [2(n+\beta) + 2\beta(n-1)\mu]\ [|a_n| + |b_n|](r-p)^2 r^n \\ &- 2\alpha(1-\beta)(r-p) + 2\alpha\mu(2r-p)(1-\beta) \qquad (7) \end{aligned}$$

The inequality in (7) holds true for all r $(0 \leq r < 1)$. Therefore letting $r \to 1$ in (7), we obtain

$$(1-p)^2 M(r) \leq \sum_{n=1}^{\infty} 2n^k[(n+\beta)+\beta(n-1)\mu]\ (1-p)^2[|a_n|+|b_n|] \\ -2\alpha(1-\beta)(1-p)+2\alpha\mu(2-p)(1-\beta).$$

By the hypothesis (6), it follows that (5) holds, so that $f \in HS_p^*(k,\alpha,\beta,\mu)$. We observe that f is sense-preserving in $\Delta\backslash\{p\}$. This is because

$$\begin{aligned}
|h'(z)| &\geq \frac{1}{|z-p|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\
&\geq \frac{1}{|z|^2} - \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \\
&\geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n|r^{n-1} \\
&\geq 1 - \sum_{n=1}^{\infty} n|a_n| \\
&\geq 1 - \sum_{n=1}^{\infty} n[(n+\beta)+\beta(n-1)\mu](1-p)^2|a_n| \\
&\geq \sum_{n=1}^{\infty} n[(n+\beta)+\beta(n-1)\mu](1-p)^2|b_n| \\
&\geq \sum_{n=1}^{\infty} n|b_n| \geq \sum_{n=1}^{\infty} n|b_n|\ |z|^{n-1} \geq |g'(z)|.
\end{aligned}$$

Hence the theorem.

Letting $k = \beta = 0$ and $p \to 0$ in Theorem 1, then we have the next corollary:

Corollary 2. *If $f(z) = h(z) + \overline{g(z)}$ is of the form (2) and satisfies the condition*

$$\sum_{n=1}^{\infty} n(|a_n|+|b_n|) \leq \alpha(1-2\mu)$$

then f is sense preserving in $\Delta\backslash\{0\}$ and $f \in HS_0^*(0,\alpha,0,\mu)$.

Remark 2. *Let $k = \beta = \mu = 0$ and $p \to 0$ in Theorem 1, then we have a result obtained by Ghanim and Darus [6].*

Remark 3. *Let $\mu = k = \beta = 0$, $\alpha = 1$ and $p \to 0$ in Theorem 1, then we have a result obtained by Bostanci, Yalcin and Öztürk [2].*

Letting $k = 1$, $\beta = 0$ and $p \to 0$ in Theorem 1, then we have the next corollary:

Corollary 3. *If $f(z) = h(z) + \overline{g(z)}$ is of the form (2) and satisfies the condition*

$$\sum_{n=1}^{\infty} n^2(|a_n| + |b_n|) \le \alpha(1 - 2\mu)$$

then f is sense preserving in $\Delta\backslash\{0\}$ and $f \in HS_0^*(1, \alpha, 0, \mu)$.

Remark 4. *Let $k = 1$, $\beta = 0$, $\alpha = 1$, $\mu = 0$ and $p \to 0$ in Theorem 1, then we have a result due to Bostanci, Yalcin and Öztürk [2].*

Next we obtain a necessary and sufficient condition for a function $f \in \overline{HS}_p$ given by (3) to be in $HS_p^*[k, \alpha, \beta, \mu]$.

Theorem 4. *Let $f \in \overline{HS}_p$ be given by (3). Then $f \in HS_p^*[k, \alpha, \beta, \mu]$ if and only if*

$$\sum_{n=1}^{\infty} n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2(a_n+b_n) \le \alpha(1-\beta)[(1-p)-\mu(2-p)] \quad (k \in N_0) \quad (8)$$

is satisfied. The estimate (8) is sharp and the equality is attained for the function

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)[(1-p)-\mu(2-p)]}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2} z^n + \frac{\alpha(1-\beta)[(1-p)-\mu(2-p)]}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2} \overline{z}^n.$$

Proof. The if part follows from Theorem 1. Hence, it suffices to show that the 'only if' part is true.

Assume that $f \in HS_p^*[k, \alpha, \beta, \mu]$. Then

$$\begin{aligned}
&\left| \frac{\dfrac{H(z)}{\mu H(z) + (1-\mu)} + 1}{\dfrac{H(z)}{\mu H(z) + (1-\mu)} + 2\beta - 1} \right| \\
&= \left| \frac{\dfrac{-\alpha p - \alpha\mu(2z-p)}{(z-p)^2} + \displaystyle\sum_{n=1}^{\infty} n^k[(n+1)+\mu(n-1)](a_n z^n + \overline{b_n z^n})}{\dfrac{-2\alpha z + 2\alpha\beta z - 2\alpha\beta p + \alpha p - \mu[4\alpha\beta z - 2\alpha z - 2\alpha\beta p + \alpha p]}{(z-p)^2} + \displaystyle\sum_{n=1}^{\infty} n^k[(n+2\beta-1)+\mu(2\beta-1)(n-1)]\,[a_n z^n + \overline{b_n z^n}]} \right| \\
&\le 1, \qquad (9)
\end{aligned}$$

$z \in \Delta \backslash \{p\}$.
Since $Re(z) \leq |z|$ for all z it follows from (9) that

$$Re \left\{ \frac{\dfrac{-\alpha p - \alpha\mu(2z-p)}{(z-p)^2} + \displaystyle\sum_{n=1}^{\infty} n^k[(n+1)+\mu(n-1)](a_n z^n + \overline{b_n z^n})}{\dfrac{-2\alpha[(1-\beta)z+\beta p] + \alpha p - \mu[4\alpha\beta z - 2\alpha z - 2\alpha\beta p + \alpha p]}{(z-p)^2} + \displaystyle\sum_{n=1}^{\infty} n^k[(n+2\beta-1)+\mu(2\beta-1)(n-1)]\ [a_n z^n + \overline{b_n z^n}]} \right\}$$
$$\leq 1,\ z \in \Delta \backslash \{p\}. \tag{10}$$

Choosing the values z on the real axis and upon clearing the denominator in (10) and letting $z \to 1$ through real values, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^k[(n+1)+\mu(n-1)](1-p)^2(a_n+b_n) \\
&\leq 2\alpha(1-\beta)(1-p) - 2\alpha\mu(2-p)(1-\beta) \\
&- \sum_{n=1}^{\infty} n^k[(n+2\beta-1)+\mu(2\beta-1)(n-1)](1-p)^2(a_n+b_n)
\end{aligned}$$

which immediately yields the required condition (8).

3. Distortion Theorem

We now prove the following distortion theorem for functions in the class $HS_p^*[k,\alpha,\beta,\mu]$.

Theorem 5. *If the function f defined by (3) is in the class $HS_p^*[k,\alpha,\beta,\mu]$, then for $|z| = r$, we have*

$$|f(z)| \leq \frac{\alpha}{r-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} r$$

Proof. Let $f \in HS_p^*[k, \alpha, \beta, \mu]$, taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &\leq \frac{\alpha}{r-p} + \sum_{n=1}^{\infty} (a_n + b_n) r^n \\
&\leq \frac{\alpha}{r-p} + \frac{[\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)]}{(1+\beta)(1-p)^2} \\
&\qquad \sum_{n=1}^{\infty} n^k \frac{[(n+\beta) + \beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} (a_n + b_n) r \\
&\leq \frac{\alpha}{r-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} r.
\end{aligned}$$

The functions

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} z$$

and

$$f(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{(1+\beta)(1-p)^2} \overline{z}$$

for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, $0 \leq \mu < 1$ show that the bound given in Theorem 5 are sharp in $\Delta \backslash \{p\}$.

Theorem 6. *Let*

$$h_0(z) = \frac{\alpha}{z-p}, \qquad g_0(z) = 0,$$

for $n = 1, 2, 3, \ldots,$

$$h_n(z) = \frac{\alpha}{z-p} + \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} z^n \tag{11}$$

and

$$g_n(z) = \frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta) + \beta(n-1)\mu](1-p)^2} \overline{z}^n \tag{12}$$

Then $f \in HS_p^*[k, \alpha, \beta, \mu]$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} (\lambda_n h_n + \gamma_n g_n) \tag{13}$$

where $\lambda_n \geq 0$, $\gamma_n \geq 0$ *and* $\sum_{n=0}^{\infty} (\lambda_n + \gamma_n) = 1$. *In particular, the extreme points of* $HS_p^*[k, \alpha, \beta, \mu]$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. From (11), (12) and (13), we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}(\lambda_n h_n + \gamma_n g_n) \\ &= \sum_{n=0}^{\infty}(\lambda_n + \gamma_n)\frac{\alpha}{z-p} + \sum_{n=1}^{\infty}\frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2}\lambda_n z^n \\ &\quad + \sum_{n=0}^{\infty}\frac{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2}\gamma_n \overline{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty} n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2 \frac{\lambda_n}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2} \\ &+ \sum_{n=0}^{\infty} n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2 \frac{\gamma_n}{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2} \\ &= \sum_{n=1}^{\infty}(\lambda_n + \gamma_n) - \lambda_0 = 1 - \lambda_0 \leq 1 \end{aligned}$$

So $f \in HS_p^*[k, \alpha, \beta, \mu]$.
Conversely, suppose that $f \in HS_p^*[k, \alpha, \beta, \mu]$. Set

$$\lambda_n = \frac{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} a_n, \quad n \geq 1$$

and

$$\gamma_n = \frac{n^k[(n+\beta)+\beta(n-1)\mu](1-p)^2}{\alpha(1-\beta)(1-p) - \alpha\mu(2-p)(1-\beta)} b_n, \quad n \geq 0$$

Then, by Theorem 4, $0 \leq \lambda_n \leq 1$ $(n = 1, 2, \dots)$ and $0 \leq \gamma_n \leq 1$, $(n = 0, 1, 2, \dots)$.
Define

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n - \sum_{n=0}^{\infty} \gamma_n$$

and note that, by Theorem 4, $\lambda_0 \geq 0$.
Consequently, we obtain

$$f(z) = \sum_{n=0}^{\infty}(\lambda_n h_n + \gamma_n g_n),$$

as required.

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